ARTICLE NO. 80-4011

Explicit High-Order Finite-Difference Analysis of Rotationally Symmetric Shells

Troy Alvin Smith*
U.S. Army Missile Command, Redstone Arsenal, Ala.

The system of equations for the analysis of rotationally symmetric shells subjected to time-dependent loadings and boundary conditions has been formulated with the transverse, meridional, and circumferential displacements as the dependent variables in the field equations. Inertia forces are considered in each of the three coordinate directions of the shell. The solution for each Fourier harmonic is obtained by employing ordinary finite-difference representations of the time derivatives and high-order finite-difference representations for the meridional coordinate derivatives. The complete system of equations is solved implicitly for the first time increment, while an explicit solution for variables within the boundary edges of the shell, together with separate implicit solutions at each boundary, is utilized for the second and succeeding time increments. To aid in the choice of a time increment, the equations for the three lower frequencies of vibration of the shell for each Fourier component of response have been derived by the Rayleigh-Ritz method. The developed equations have been programmed in FORTRAN IV language to provide solutions for general shell geometries and loadings by electronic computer. Solutions obtained with the program for typical shells and loadings have been found to be stable and in agreement for a wide range of practical values of both spatial and time increments. Solutions for a typical shell and loading together with comparison of the stability requirements with the stability requirements for other formulations and finite-difference representations have been included.

Nomenclature

$A_1,\ldots,A_{10}$ = parameters used in Eq. (25) and defined in Ref. 5
$B_1,\ldots,B_{13}$ = parameters used in Eq. (26) and defined in Ref. 5
$C_1,\ldots,C_9$ = parameters used in Eq. (27) and defined in Ref. 5
C = coefficients of the force variables $N_{\phi n}$, $M_{\phi n}$, N_n, and Q_n in the finite-difference equations obtained before change of force variables
C^0 = coefficients of the modified force variables $N^0_{\phi n}$, $M^0_{\phi n}$, N^0_n, and Q^0_n in the governing finite-difference equations
$D_1,\ldots,D_{50}$ = parameters used in Eqs. (29-34) and defined in Ref. 5
D = $Eh^3/12(1-\nu^2)$, flexural rigidity of the shell
E = Young's modulus
g = acceleration constant
h = thickness of the shell
K = $Eh/(1-\nu^2)$, extensional rigidity of the shell
m_θ, m_ϕ = moments of the mechanical surface loads
$M_\theta, M_\phi, M_{\theta\phi}$ = moment stress resultants
$M^0_{\phi n}$ = $M_{\phi n}\times 10^{-6}$
n = integer, designating the nth Fourier component
N,Q = effective shear resultants
N^0_n, Q^0_n = $N_n \times 10^{-6}$ and $Q_n\times 10^{-6}$
$N_\theta, N_\phi, N_{\theta\phi}$ = membrane stress resultants
$N^0_{\phi n}$ = $N_{\phi n}\times 10^{-6}$
p, p_θ, p_ϕ = components of the mechanical surface loads
Q_θ, Q_ϕ = transverse shear resultants
r = distance of point on the middle surface of the shell from the axis of symmetry
R_θ, R_ϕ = principal radii of curvature of the middle surface of the shell
s = distance from an arbitrary origin along the meridian of the shell in the positive direction of ϕ
Δs = increment of the space variable s
s_0 = value of the coordinate s at the boundary s_0 (denoted also as the boundary z_0) of the shell
s_N = value of the coordinate s at the boundary s_N (denoted also as the boundary z_N) of the shell
s_i = point on meridional line of the shell at station i, where i varies consecutively from $i=-1$ to $i=N+1$
t = independent time variable
Δt = increment of the time variable t
t_0 = initial value of the time variable t
t_1 = value of the time variable one time increment after time t_0
T, T_0, T_1 = temperature increment and temperature resultants
u_θ, u_ϕ, w = components of displacement of the middle surface of the shell
$\dot{u}_\theta, \dot{u}_\phi, \dot{w}$ = components of velocity of the middle surface of the shell
z = distance of point on the middle surface of the shell measured from the origin along the axis of symmetry
z_0 = value of the coordinate z at the boundary z_0 (denoted also as the boundary s_0) of the shell
z_N = value of the coordinate z at the boundary z_N (denoted also as the boundary s_N) of the shell
α = coefficient of thermal expansion of shell material
β_θ, β_ϕ = angles of rotation of the normal to the middle surface of the shell

Received Aug. 21, 1978; revision received Feb. 2, 1979.

Index categories: Structural Statics; Analytical and Numerical Methods; Structural Dynamics.

*Aerospace Engineer.

γ = weight of shell material per unit volume
θ,ϕ,ρ = coordinates of any point of the shell
μ = parameter used in Eq. (50) to select an increment Δt
ν = Poisson's ratio
ω = circular frequency of vibration of the shell, rad/s
ω_{max} = highest calculated circular frequency of vibration of the shell

Introduction

In the absence of closed-form solutions for the general shell problem, several investigators have obtained solutions by numerical methods. These investigators include those mentioned by Smith[1] in a 1971 report in which numerical procedures were given for determining the response of rotationally symmetric open-ended thin shells of revolution under time-dependent impulsive and thermal loadings. In that report, inertia forces were considered in directions normal to the middle surface and along the meridians of the shell. The field equations consisted of eight first-order partial differential equations with respect to the meridional coordinate of the shell, and the solution for each Fourier harmonic was obtained by employing low-order finite-difference representations for both time and spatial derivatives. The fundamental variables were determined implicitly for the first time increment, while explicit relations were used to obtain displacements normal to the middle surface and along the meridians of the shell with the exception of quantities on and in the near vicinity of each boundary for the second and later time increments. Numerically stable solutions for typical examples were found for a wide range of practical values of both spatial and time increments.

In 1972, Smith[2] published numerical procedures for computing the response of thin conical shells under time-dependent axisymmetric impulsive and thermal loadings. Inertia forces were considered in directions normal to the middle surface and along the meridians of the shell. The field equations consisted of one third-order, one second-order, and one first-order partial differential equation with respect to the meridional coordinate of the shell. Solutions were obtained by using low-order finite-difference representations for time derivatives and employing higher-order finite-difference representations for the spatial derivatives. The fundamental variables were evaluated implicitly for the first time increment, while explicit relations were used for the three fundamental variables with the exception of quantities on and in the near vicinity of each boundary for the second and later time increments. Numerically stable solutions were found for typical examples for a wide range of practical values of time and spatial increments.

In 1973, Smith[3] published numerical procedures for evaluating the response of rotationally symmetric open-ended thin shells of revolution under time-dependent surface and thermal loadings using a higher-order finite-difference representation of spatial derivatives than that used in Ref. 1. The governing differential equations were the same as in Ref. 1 with the exception that surface loadings and inertia forces were considered in all three coordinate directions of the shell. Initial attempts to obtain explicit solutions for the second and later time increments resulted in oscillatory instabilities in the numerical solutions for a considerable range of values of spatial and time increments, including impractically small values of the time increment, for all typical solutions investigated. Consequently, for the referenced report, the explicit solutions initially desired were abandoned in favor of a stable implicit solution to the complete system of equations for all time increments.

In this report, our purpose is the development of numerical procedures which, for given accuracy, permit the use of larger meridional and time increments than the procedures used in Ref. 1. The surface, thermal, and inertia forces and the order of the spatial derivative representations are the same as in Ref. 3. However, the field equations are in terms of the transverse, meridional, and circumferential displacements only as the dependent variables, thus yielding three differential equations of higher order than the eight first-order equations used in Refs. 1 and 3. This resulted in a system of equations for which explicit solutions have been found to be stable for a wide range of practical values of both spatial and time increments.

Governing Differential Equations

Our system of governing equations will be based on the linear classical theory of shells as given by Reissner.[4] Surface loadings and inertia forces in each of the three coordinate directions w, u_ϕ, and u_θ will be considered. All rotary inertia terms will be neglected. The thickness h of the shell may vary along the meridian, and we assume continuity of h and its derivatives through the second order. We assume that $\rho/R_\phi \ll 1$ and that $\rho/R_\theta \ll 1$. Hence, we take $N_{\theta\phi}=N_{\phi\theta}$ and $M_{\theta\phi}=M_{\phi\theta}$.

The geometry and coordinate system for the middle surface of our shell is shown in Fig. 1. Shell element membrane and shear forces are shown in Fig. 2, and shell element bending and twisting moments are shown in Fig. 3. The geometry of the middle surface will be defined by the coordinate z. However, we will develop our equations with meridional coordinate s as an independent variable. The function $r=r(z)$ defines the geometry of the middle surface, and the principal radii of curvature of this surface may be expressed as

$$R_\phi = -[1+(r_{,z})^2]^{3/2}/r_{,zz} \tag{1}$$

$$R_\theta = r[1+(r_{,z})^2]^{1/2} \tag{2}$$

The required derivatives of R_ϕ with respect to the coordinate s are given in Ref. 5. To account for variation of temperature through the thickness, it is convenient to introduce temperature resultants found by integrating the temperature distribution through the thickness. In accordance with Ref. 1, these resultants are

$$T_0(\theta,s,t) = \frac{1}{h}\int_{-h/2}^{h/2} T(\theta,s,\rho,t)\,\mathrm{d}\rho \tag{3}$$

$$T_1(\theta,s,t) = \frac{12}{h^3}\int_{-h/2}^{h/2} \rho T(\theta,s,\rho,t)\,\mathrm{d}\rho \tag{4}$$

To maintain the advantages of uncoupled Fourier series solutions for nonsymmetrical thermal loadings and to maintain simplification of procedures for axisymmetric thermal loadings, we assume that

$$E=\text{const},\quad \nu=\text{const},\quad \alpha=\text{const} \tag{5}$$

We define the quantities w, u_ϕ, u_θ, β_ϕ, Q, N_ϕ, N, and M_ϕ to constitute the primary variables in our system of equations. The variables β_θ, N_θ, $N_{\theta\phi}$, M_θ, $M_{\theta\phi}$, Q_ϕ, and Q_θ are designated as the secondary variables.

From two of our five useful equations of equilibrium for a typical element of the shell, we find the shear resultants Q_ϕ and Q_θ to be

$$Q_\phi = (1/r)M_{\theta\phi,\theta} + M_{\phi,s} + (\cos\phi/r)(M_\phi - M_\theta) + m_\phi \tag{6}$$

$$Q_\theta = (1/r)M_{\theta,\theta} + M_{\theta\phi,s} + (2\cos\phi/r)M_{\theta\phi} + m_\theta \tag{7}$$

By substituting Eqs. (6) and (7) into the remaining three equations of equilibrium, we find our three field equations to

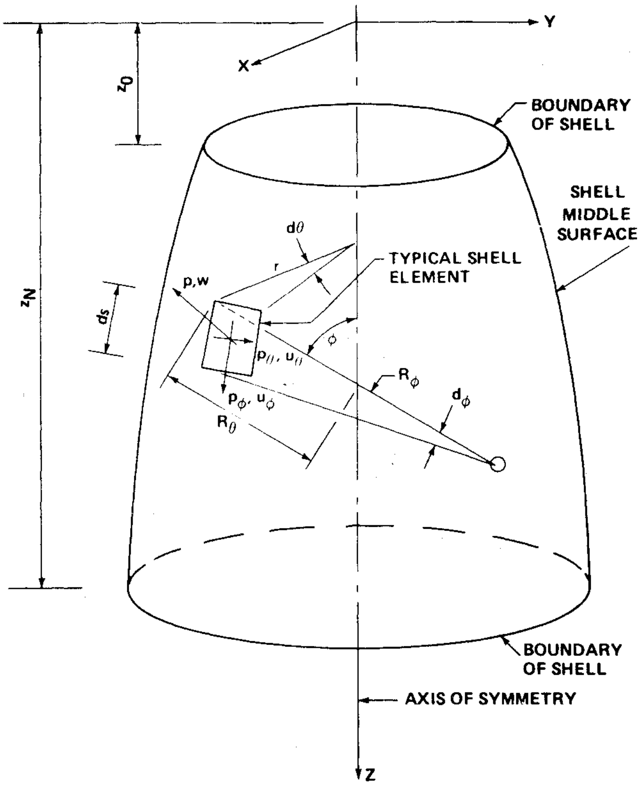

Fig. 1 Typical shell of revolution.

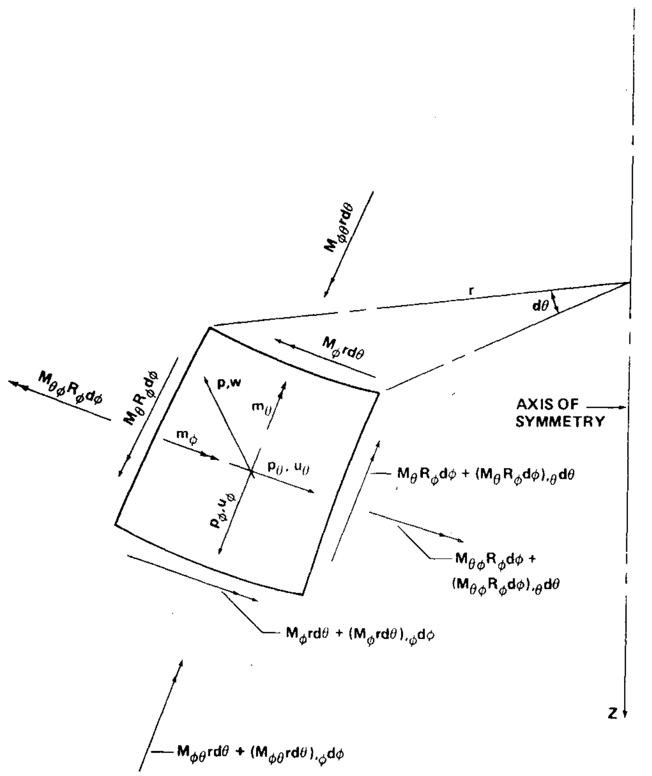

Fig. 3 Shell element bending and twisting moments.

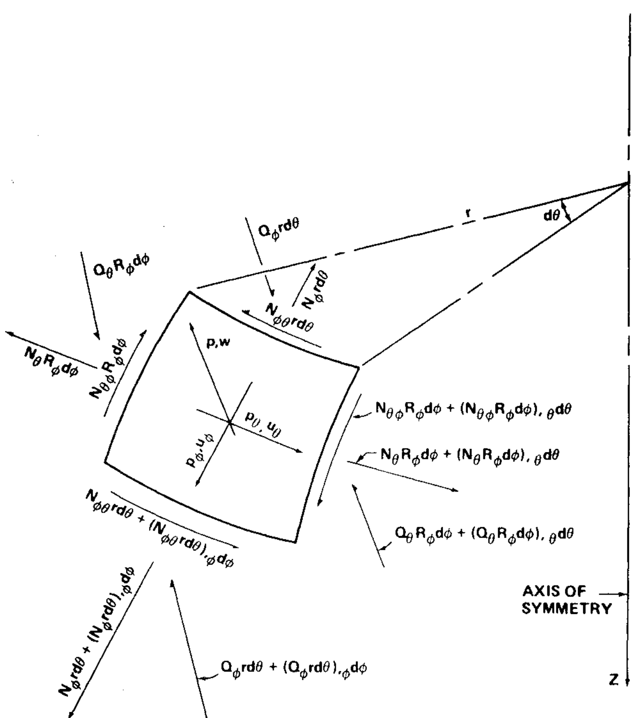

Fig. 2 Shell element membrane and shear forces.

be satisfied within the boundary edges of the shell to be

$$N_{\theta\phi,\theta}+rN_{\phi,s}+(N_\phi-N_\theta)\cos\phi+\frac{1}{R_\phi}M_{\theta\phi,\theta}+\frac{r}{R_\phi}M_{\phi,s}$$

$$+\frac{\cos\phi}{R_\phi}(M_\phi-M_\theta)+r\left(\frac{m_\phi}{R_\phi}+p_\phi\right)-\frac{\gamma hr}{g}u_{\phi,tt}=0 \tag{8}$$

$$\frac{1}{r}M_{\theta,\theta\theta}+2M_{\theta\phi,\theta s}+\frac{2\cos\phi}{r}M_{\theta\phi,\theta}+rM_{\phi,ss}$$

$$+2\cos\phi M_{\phi,s}-\frac{\sin\phi}{R_\phi}(M_\phi-M_\theta)-\cos\phi M_{\theta,s}$$

$$-N_\theta\sin\phi-\frac{r}{R_\phi}N_\phi+r(m_{\phi,s}+p)+\cos\phi m_\phi$$

$$+m_{\theta,\theta}-\frac{\gamma hr}{g}w_{,tt}=0 \tag{9}$$

$$N_{\theta,\theta}+rN_{\theta\phi,s}+2\cos\phi N_{\theta\phi}+\frac{\sin\phi}{r}M_{\theta,\theta}$$

$$+\sin\phi M_{\theta\phi,s}+\frac{2\sin\phi\cos\phi}{r}M_{\theta\phi}+\sin\phi m_\theta$$

$$+rp_\theta-\frac{\gamma hr}{g}u_{\theta,tt}=0 \tag{10}$$

By using the stress-strain and strain-displacement relations given by Kalnins,[6] we find the stress resultants in terms of the displacements w, u_ϕ, and u_θ to be

$$N_\theta=K\left[\frac{1}{r}(u_{\theta,\theta}+u_\phi\cos\phi+w\sin\phi)\right.$$

$$\left.+\nu\left(u_{\phi,s}+\frac{w}{R_\phi}\right)\right]-(1+\nu)\alpha KT_0 \tag{11}$$

$$N_\phi=K\left[u_{\phi,s}+\frac{w}{R_\phi}+\frac{\nu}{r}(u_{\theta,\theta}+u_\phi\cos\phi+w\sin\phi)\right]$$

$$-(1+\nu)\alpha KT_0 \tag{12}$$

$$N_{\theta\phi}=\frac{(1-\nu)K}{2}\left[\frac{1}{r}(u_{\phi,\theta}-u_\theta\cos\phi)+u_{\theta,s}\right] \qquad (13)$$

$$M_\theta=D\Big[-\frac{1}{r^2}w,_{\theta\theta}+\frac{\sin\phi}{r^2}u_{\theta,\theta}+\frac{\cos\phi}{r}\left(-w,_s+\frac{1}{R_\phi}u_\phi\right)$$
$$+\nu\left(-w,_{ss}+\frac{1}{R_\phi}u_{\phi,s}-\frac{1}{R_\phi^2}R_{\phi,s}u_\phi\right)\Big]-(1+\nu)\alpha DT_1 \qquad (14)$$

$$M_\phi=D\Big[-w,_{ss}+\frac{1}{R_\phi}u_{\phi,s}-\frac{1}{R_\phi^2}R_{\phi,s}u_\phi$$
$$+\frac{\nu}{r}\left(-\frac{1}{r}w,_{\theta\theta}+\frac{\sin\phi}{r}u_{\theta,\theta}-\cos\phi w,_s+\frac{\cos\phi}{R_\phi}u_\phi\right)\Big]$$
$$-(1+\nu)\alpha DT_1 \qquad (15)$$

$$M_{\theta\phi}=\frac{(1-\nu)D}{2r}\Big[-2w,_{\theta s}+\frac{2\cos\phi}{r}w,_\theta+\frac{1}{R_\phi}u_{\phi,\theta}$$
$$+\sin\phi u_{\theta,s}+\left(\frac{\cos\phi}{R_\phi}-\frac{2\sin\phi\cos\phi}{r}\right)u_\theta\Big] \qquad (16)$$

The quantities N and Q are the effective shear resultants defined as

$$N=N_{\theta\phi}+(\sin\phi/r)M_{\theta\phi} \qquad (17)$$

$$Q=Q_\phi+(1/r)M_{\theta\phi,\theta} \qquad (18)$$

The rotations of the normals to the middle surface are

$$\beta_\phi=-(1/R_\phi)w,_\phi+(1/R_\phi)u_\phi \qquad (19)$$

$$\beta_\theta=-(1/r)w,_\theta+(\sin\phi/r)u_\theta \qquad (20)$$

We obtain our three field equations in terms of the displacements w, u_ϕ, and u_θ by substituting the stress resultants defined by Eqs. (11-16) together with their appropriate derivatives into Eqs. (8-10). These rather lengthy and cumbersome equations may be found in Ref. 5.

In the classical theory of shells, the quantities which appear in the natural boundary conditions on a rotationally symmetric edge of a shell of revolution are the generalized displacements w, u_ϕ, u_θ, and β_ϕ and the generalized forces Q, N_ϕ, N, and M_ϕ. To incorporate the remaining natural boundary values β_ϕ, Q, N_ϕ, N, and M_ϕ into our system of equations as variables at the boundary edges of the shell, we supplement our three field equations in terms of the displacements w, u_ϕ, and u_θ by writing Eqs. (19, 18, 12, 17, and 15) for β_ϕ, Q, N_ϕ, N, and M_ϕ, respectively, at each boundary in terms of the displacements w, u_ϕ, and u_θ. The three field equations, together with the definition of β_ϕ, Q, N_ϕ, N, and M_ϕ at each boundary, will be solved in conjunction with the equations defining the boundary conditions and the initial conditions. For the boundary conditions, we will prescribe the appropriate four of the quantities w, u_ϕ, u_θ, β_ϕ, Q, N_ϕ, N, and M_ϕ at each boundary. Thus, the boundary conditions to be considered at the boundary s_0 are

$$w(s_0,\theta,t)=w'(s_0,\theta,t) \text{ or } Q(s_0,\theta,t)=Q'(s_0,\theta,t) \qquad (21a)$$

$$u_\phi(s_0,\theta,t)=u_\phi'(s_0,\theta,t) \text{ or } N_\phi(s_0,\theta,t)=N_\phi'(s_0,\theta,t) \qquad (21b)$$

$$u_\theta(s_0,\theta,t)=u_\theta'(s_0,\theta,t) \text{ or } N(s_0,\theta,t)=N'(s_0,\theta,t) \qquad (21c)$$

$$\beta_\phi(s_0,\theta,t)=\beta_\phi'(s_0,\theta,t) \text{ or } M_\phi(s_0,\theta,t)=M_\phi'(s_0,\theta,t) \qquad (21d)$$

where the primed variables indicate specified values of the respective quantities. Similar boundary conditions will be imposed at the boundary s_N.

For the initial conditions, we will prescribe initial values of the displacements and velocities in each of the coordinate directions w, u_ϕ, and u_θ. Thus, the initial conditions to be considered are typically

$$w(s,\theta,t_0)=w'(s,\theta,t_0) \qquad (22a)$$

$$\dot{w}(s,\theta,t_0)=\dot{w}'(s,\theta,t_0) \qquad (22b)$$

where the primed variables indicate specified values of the initial displacements and velocities.

Upon solution of the system of equations consisting of the field equations, the equations defining β_ϕ, Q, N_ϕ, N, and M_ϕ at each boundary, and the equations for the imposed boundary and initial conditions, values for the primary variables β_ϕ, Q, N_ϕ, N, and M_ϕ and for the secondary variables β_θ, N_θ, $N_{\theta\phi}$, M_θ, $M_{\theta\phi}$, Q_ϕ, and Q_θ may be evaluated from the equations already given for these variables.

To solve our system of equations, we expand all loadings and dependent variables in the circumferential direction of the shell in Fourier series. We will truncate these infinite series at a finite number of terms for the solution of specific shell problems.

The Fourier series representations of the loadings p_ϕ, m_ϕ, p, T_0, and T_1, the primary variables w, u_ϕ, β_ϕ, Q, N_ϕ, and M_ϕ, and the secondary variables N_θ, M_θ, and Q_ϕ are typically

$$p_\phi=\sum_{n=0}^{P}p_{\phi n}(s,t)\cos n\theta+\sum_{n=1}^{P}\bar{p}_{\phi n}(s,t)\sin n\theta \qquad (23)$$

The loadings p_θ and m_θ, the primary variables u_θ and N, and the secondary variables β_θ, $N_{\theta\phi}$, $M_{\theta\phi}$, and Q_θ are typically

$$p_\theta=\sum_{n=1}^{P}p_{\theta n}(s,t)\sin n\theta+\sum_{n=0}^{P}\bar{p}_{\theta n}(s,t)\cos n\theta \qquad (24)$$

Upon substituting these Fourier series representations into our system of equations, we obtain $P+1$ separate decoupled systems of equations in the variables s and t to solve in lieu of the single system of equations in the variables θ, s, and t. For each system we obtain two separate sets of equations—one for the variables which are designated without a bar and another for the variables which are designated with a bar. Here and elsewhere in the sequel where double signs occur in the equations, the upper sign is to accompany the first set of equations and the lower sign is to apply to the second set. Single signs will apply to both sets.

To facilitate the finite-difference solution of our equations, we substitute subscripted alphabetical coefficients and loading terms for the more lengthy coefficients and loading terms in the equations. These are the quantities A_1-A_{10}, B_1-B_{13}, C_1-C_9, and D_1-D_{50}, which involve geometric and material parameters, loading terms, and the Fourier component designator n. These quantities may be found in Ref. 5.

With the aforementioned definition of coefficients and loading terms, our field equations for each Fourier harmonic are

$$-A_1w_{n,sss}-A_2w_{n,ss}+A_3w_{n,s}+A_4w_n$$
$$+A_5u_{\phi n,ss}+A_6u_{\phi n,s}+A_7u_{\phi n}\pm A_8u_{\theta n,s}\pm A_9u_{\theta n}$$
$$-(\gamma hr/g)u_{\phi n,tt}=A_{10}-r[p_{\phi n}+(1/R_\phi)m_{\phi n}] \qquad (25)$$

$$-B_1w_{n,ssss}-B_2w_{n,sss}+B_3w_{n,ss}+B_4w_{n,s}+B_5w_n$$
$$+B_6u_{\phi n,sss}+B_7u_{\phi n,ss}+B_8u_{\phi n,s}+B_9u_{\phi n}$$

$$\pm B_{10}u_{\theta n,ss} \pm B_{11}u_{\theta n,s} \pm B_{12}u_{\theta n} - (\gamma hr/g)\,w_{n,tt}$$
$$= B_{13} \mp nm_{\theta n} - r(p_n + m_{\phi n,s}) - m_{\phi n}\cos\phi \quad (26)$$

$$\pm C_1 w_{n,ss} \pm C_2 w_{n,s} \mp C_3 w_n \mp C_4 u_{\phi n,s} \mp C_5 u_{\phi n}$$
$$+ C_6 u_{\theta n,ss} + C_7 u_{\theta n,s} + C_8 u_{\theta n} - (\gamma hr/g)\,u_{\theta n,tt}$$
$$= \mp C_9 - m_{\theta n}\sin\phi - rp_{\theta n} \quad (27)$$

The primary variables $\beta_{\phi n}$, Q_n, $N_{\phi n}$, N_n, and $M_{\phi n}$ are given by

$$\beta_{\phi n} = -w_{n,s} + (1/R_\phi)\,u_{\phi n} \quad (28)$$

$$Q_n = -Dw_{n,sss} - D_{42}w_{n,ss} + D_{43}w_{n,s} - D_{44}w_n + D_{45}u_{\phi n,ss}$$
$$+ D_{46}u_{\phi n,s} + D_{47}u_{\phi n} \pm D_{48}u_{\theta n,s} \pm D_{49}u_{\theta n} - D_{50} + m_{\phi n} \quad (29)$$

$$N_{\phi n} = K(D_1 w_n + u_{\phi n,s} + D_2 u_{\phi n} \pm D_3 u_{\theta n} - D_4 T_{0n}) \quad (30)$$

$$N_n = \pm D_{37}w_{n,s} \mp D_{38}w_n \mp D_{39}u_{\phi n} + D_{40}u_{\theta n,s} + D_{41}u_{\theta n} \quad (31)$$

$$M_{\phi n} = D[-w_{n,ss} - D_2 w_{n,s} + D_5 w_n + (1/R_\phi)\,u_{\phi n,s}$$
$$+ D_6 u_{\phi n} \pm D_7 u_{\theta n} - D_4 T_{1n}] \quad (32)$$

It will be convenient to also express the primary variables N_n and Q_n as

$$N_n = N_{\theta\phi n} + D_{36}M_{\theta\phi n} \quad (33)$$

$$Q_n = Q_{\phi n} \pm D_{10}M_{\theta\phi n} \quad (34)$$

The boundary conditions to be considered at the boundary s_0 for each Fourier harmonic are

$$w_n(s_0,t) = w_n'(s_0,t) \text{ or } Q_n(s_0,t) = Q_n'(s_0,t) \quad (35a)$$

$$u_{\phi n}(s_0,t) = u_{\phi n}'(s_0,t) \text{ or } N_{\phi n}(s_0,t) = N_{\phi n}'(s_0,t) \quad (35b)$$

$$u_{\theta n}(s_0,t) = u_{\theta n}'(s_0,t) \text{ or } N_n(s_0,t) = N_n'(s_0,t) \quad (35c)$$

$$\beta_{\phi n}(s_0,t) = \beta_{\phi n}'(s_0,t) \text{ or } M_{\phi n}(s_0,t) = M_{\phi n}'(s_0,t) \quad (35d)$$

where the primed variables denote specified quantities. Similar boundary conditions will be imposed at the boundary s_N.

The initial conditions for each Fourier harmonic are typically

$$w_n(s,t_0) = w_n'(s,t_0) \quad (36a)$$

$$\dot{w}_n(s,t_0) = \dot{w}_n'(s,t_0) \quad (36b)$$

where the primed variables are specified quantities.

The equations for the Fourier components of the secondary variables, together with the details of the derivation of our governing equations for each Fourier harmonic, may be found in Ref. 5.

Conversion of Equations to Finite-Difference Form

Our equations to be solved for each Fourier harmonic consist of Eqs. (25-27) applied on the interval $s_0 \le s \le s_N$, Eqs. (28-32) evaluated at each boundary, Eqs. (35) for the boundary conditions, and Eqs. (36) for the initial conditions. When this system of equations has been solved, the variables $\beta_{\phi n}$, Q_n, $N_{\phi n}$, N_n, and $M_{\phi n}$ may be determined from Eqs. (28-32), respectively. To solve this system of equations, we replace all derivatives with their finite-difference equivalents to obtain a system of algebraic equations which may be applied at successive increments of the time variable.

Our accelerations at time t_1 are typically

$$\ddot{w}_n(s,t_1) = \frac{2[w_n(s,t_1) - w_n(s,t_0) - (\Delta t)\,\dot{w}_n(s,t_0)]}{(\Delta t)^2} \quad (37)$$

For times $t \ge t_0 + 2\Delta t$, accelerations will be typically given by

$$\ddot{w}_n(s,t-\Delta t) = \frac{w_n(s,t-2\Delta t) - 2w_n(s,t-\Delta t) + w_n(s,t)}{(\Delta t)^2} \quad (38)$$

With the exception of spatial derivatives on the boundaries, we neglect terms of the fourth and higher powers of Δs in the power series expansions for the derivatives. For derivatives on the boundaries, we neglect terms of the fifth and higher powers of Δs. Wherever it is possible, we use a central-difference representation for all derivatives. For derivatives at points on and near the boundaries, we use finite-difference representations which are unbalanced about the pivotal points. Derivatives of $u_{\phi n}$ and $u_{\theta n}$ will be written in terms of these variables at points on and inside the boundaries. To enable us to specify $\beta_{\phi n}$ at the boundaries and to equalize the number of variables and the number of equations, we include one fictitious point beyond each boundary for writing the derivatives of w_n. Thus, we have $3N+15$ variables to be determined from the $3N+15$ finite-difference equations, where N is the number of equal increments along the meridian of the shell. Our central-difference representations are typically

$$w_{n,s}(s) = \frac{1}{\Delta s}\Big[\frac{1}{12}w_n(s-2\Delta s) - \frac{2}{3}w_n(s-\Delta s)$$
$$+ \frac{2}{3}w_n(s+\Delta s) - \frac{1}{12}w_n(s+2\Delta s)\Big] \quad (39)$$

$$w_{n,ss}(s) = \frac{1}{(\Delta s)^2}\Big[-\frac{1}{12}w_n(s-2\Delta s) + \frac{4}{3}w_n(s-\Delta s)$$
$$- \frac{5}{2}w_n(s) + \frac{4}{3}w_n(s+\Delta s) - \frac{1}{12}w_n(s+2\Delta s)\Big] \quad (40)$$

$$w_{n,sss}(s) = \frac{1}{(\Delta s)^3}\Big[\frac{1}{8}w_n(s-3\Delta s) - w_n(s-2\Delta s)$$
$$+ \frac{13}{8}w_n(s-\Delta s) - \frac{13}{8}w_n(s+\Delta s) + w_n(s+2\Delta s)$$
$$- \frac{1}{8}w_n(s+3\Delta s)\Big] \quad (41)$$

$$w_{n,ssss}(s) = \frac{1}{(\Delta s)^4}\Big[-\frac{1}{6}w_n(s-3\Delta s) + 2w_n(s-2\Delta s)$$
$$- \frac{13}{2}w_n(s-\Delta s) + \frac{28}{3}w_n(s) - \frac{13}{2}w_n(s+\Delta s)$$
$$+ 2w_n(s+2\Delta s) - \frac{1}{6}w_n(s+3\Delta s)\Big] \quad (42)$$

The derivatives of w_n at the boundary s_0 are represented by

$$w_{n,s}(s_0)=\frac{1}{\Delta s}\left[-\frac{1}{5}w_n(s_{-1})-\frac{13}{12}w_n(s_0)+2w_n(s_1)-w_n(s_2)+\frac{1}{3}w_n(s_3)-\frac{1}{20}w_n(s_4)\right] \tag{43}$$

$$w_{n,ss}(s_0)=\frac{1}{(\Delta s)^2}\left[\frac{137}{180}w_n(s_{-1})-\frac{49}{60}w_n(s_0)-\frac{17}{12}w_n(s_1)+\frac{47}{18}w_n(s_2)-\frac{19}{12}w_n(s_3)+\frac{31}{60}w_n(s_4)-\frac{13}{180}w_n(s_5)\right] \tag{44}$$

$$w_{n,sss}(s_0)=\frac{1}{(\Delta s)^3}\left[-\frac{29}{15}w_n(s_{-1})+\frac{889}{120}w_n(s_0)-\frac{58}{5}w_n(s_1)+\frac{241}{24}w_n(s_2)-\frac{17}{3}w_n(s_3)+\frac{89}{40}w_n(s_4)-\frac{8}{15}w_n(s_5)+\frac{7}{120}w_n(s_6)\right] \tag{45}$$

The derivatives of $u_{\phi n}$ and $u_{\theta n}$ on the boundary s_0 will be represented typically as

$$u_{\phi n,s}(s_0)=\frac{1}{\Delta s}\left[-\frac{137}{60}u_{\phi n}(s_0)+5u_{\phi n}(s_1)-5u_{\phi n}(s_2)+\frac{10}{3}u_{\phi n}(s_3)-\frac{5}{4}u_{\phi n}(s_4)+\frac{1}{5}u_{\phi n}(s_5)\right] \tag{46}$$

$$u_{\phi n,ss}(s_0)=\frac{1}{(\Delta s)^2}\left[\frac{203}{45}u_{\phi n}(s_0)-\frac{87}{5}u_{\phi n}(s_1)+\frac{117}{4}u_{\phi n}(s_2)-\frac{254}{9}u_{\phi n}(s_3)+\frac{33}{2}u_{\phi n}(s_4)-\frac{27}{5}u_{\phi n}(s_5)+\frac{137}{180}u_{\phi n}(s_6)\right] \tag{47}$$

Derivative representations for the boundary s_N are similar to those given for the boundary s_0. These and other representations not shown are given in Ref. 5.

To convert Eqs. (25-32) to spatial finite-difference form, we replace all derivatives by their representations typified by Eqs. (39-47). Further, to produce more nearly equal coefficients in our equations, we define new force variables to be

$$N^0_{\phi n}=N_{\phi n}\times 10^{-6} \tag{48a}$$

$$M^0_{\phi n}=M_{\phi n}\times 10^{-6} \tag{48b}$$

$$N^0_{n}=N_{n}\times 10^{-6} \tag{48c}$$

$$Q^0_{n}=Q_{n}\times 10^{-6} \tag{48d}$$

and new coefficients C^0 of the force variables to be

$$C^0(N^0_{\phi n})=C(N_{\phi n})\times 10^{6} \tag{49a}$$

$$C^0(M^0_{\phi n})=C(M_{\phi n})\times 10^{6} \tag{49b}$$

$$C^0(N^0_{n})=C(N_{n})\times 10^{6} \tag{49c}$$

$$C^0(Q^0_{n})=C(Q_{n})\times 10^{6} \tag{49d}$$

By using Eq. (37) for the time derivatives, Eqs. (35) for the boundary conditions, and Eqs. (36) for the initial conditions, we obtain an implicit solution to our equations for the first time increment.

By representing our accelerations as given typically by Eq. (38) for the second and later time increments, we obtain explicit expressions for $w_n(s,t)$, $u_{\phi n}(s,t)$, and $u_{\theta n}(s,t)$ on the interval $s_1 \leq s \leq s_{N-1}$. After using these explicit expressions, we have available nine separate equations for each boundary to evaluate implicitly the remaining variables for the time t. Our nine equations for the boundary s_0 consist of Eqs. (28-32) written in finite-difference form and evaluated at s_0 together with four equations specifying the boundary conditions. The nine equations for the boundary s_N consist of the same equations evaluated at s_N together with four equations specifying those boundary conditions. The finite-difference equations for both t_1 and t are contained in Ref. 5.

Selection of Meridional and Time Increments

To solve our system of finite-difference equations, choices must be made for the increments Δs and Δt. These increments must have magnitudes which produce numerical stability of the finite-difference solution. We expect the required relations between the time and spatial increments to be dependent upon the formulation of the differential equations and the finite-difference representations used for the derivatives. The choice of the meridional increment Δs is further limited by the requirement that it be made sufficiently small to define the significant vibration modes of the shell. In the absence of tractable procedures for determining the stability limits for general systems of finite-difference equations, we will determine choices of increments Δt which may be used with a given increment Δs to achieve numerical stability of the solution by trial.

The selection of an increment Δs will be made on the basis that the finite-difference solution for the static problem must converge to the solution of the differential equations. Since stability is divorced from our consideration of the choice of an increment Δs, we need only to choose the increment to minimize truncation and roundoff errors. Upon the basis of static solutions obtained for typical shells, it appears that accurate static solutions are obtained for meridional increments which are from one to two times the thickness h of the shell. Furthermore, selection of Δs within this range will usually account for the excitation of vibration modes of relatively short wavelengths for the dynamic shell problem. We will, therefore, generally select Δs to lie within this range for solution of our equations for the dynamic shell problem.

In choosing an increment Δt, we require that the increment be sufficiently small to define the loadings, that it be a small fraction of the period of the highest significant vibration mode excited by the loadings, and that it be of the proper magnitude to produce, in conjunction with the chosen increment Δs, a numerically stable solution to our finite-difference equations. In regard to the first consideration, we determine a suitable range of values of Δt by a study of the time functions used to define the loadings. In regard to the second consideration, we expect that the significant response of the shell will usually be governed by the first few of the lower modes. We associate with each Fourier component n a family of modes consisting of the fundamental mode and the higher modes. We expect the frequency of the fundamental mode for $n=0$ to be somewhat higher than the fundamental frequencies for the next few values of n. For some small value of n, however, we obtain a minimum value of the several fundamental frequencies. For greater values of n, we expect the magnitude of the fundamental frequency to increase with increased n. In our analysis, we will, therefore, determine by the Rayleigh-Ritz method the three lower frequencies of vibration for $n=0$ and for the highest Fourier component n used to represent the loadings and dependent variables. We will then directly consider only the one of these Fourier components which has the highest calculated frequency ω_{max}. We will choose Δt to be some small fraction μ of the shortest calculated period from the relation

$$\Delta t = 2\mu\pi / \omega_{max} \tag{50}$$

where μ is a parameter to be designated upon the basis of experience with the differential equation formulation and finite-difference representations used. In regard to stability of the solution, we will analyze our shell with both our initial choice and a second smaller choice of Δt. If the two solutions agree uniformly along the meridian at all corresponding times for the same increment Δs, we accept the solutions obtained. If the two solutions do not agree, we will choose still smaller values of Δt until two solutions are in uniform agreement. Details of the determination of the three lower frequencies of vibration for the values of n are contained in Ref. 5.

Results for Typical Cylindrical Shell

Our finite-difference equations were programmed in FORTRAN IV language and all typical solutions were obtained on the CDC 6600 computer. As an illustration of the results obtained for typical shells, we include here the analysis of a cylindrical shell with the geometry and loading shown in Fig. 4. This is the same structure and loading for which solutions are given in Ref. 1 with a lower-order spatial derivative representation and the neglect of circumferential inertia forces and in Ref. 3, in which the field equations consist of eight first-order differential equations, higher-order spatial derivatives are used, and the solutions are obtained implicitly.

We assume the initial displacements and velocities to be zero. For the boundary conditions, we assume that w, u_ϕ, u_θ, and M_ϕ are zero at s_0 and that w, N_ϕ, u_θ, and M_ϕ are zero at s_N. We assume a value of 206×10^9Pa (30×10^6 lb/in.^{2}) for E, a value of 0.786×10^{-2} kg/cm^3 (0.284 lb/in.^{3}) for γ, and a value of 0.30 for ν.

For the given conditions, only the equations containing the variables and loading terms designated without a bar enter into the solution. We use the Fourier components for $n=0$ through $n=4$. The four nonzero components of loading are $p_0 = -2192 \times 10^3$, $p_1 = -3447 \times 10^3$, $p_2 = -1462 \times 10^3$, and $p_4 = 290 \times 10^3$Pa (-318, -500, -212, and 42 lb/in.^{2}). We choose an increment Δs equal to $2h$, thus dividing the length of the cylinder into 18 increments.

Using our equations developed in Ref. 5, we obtain for the Fourier component $n=0$ values for the three lower frequencies of vibration of the shell of 24,800 rad/s; 37,600 rad/s; and 44,000 rad/s. For the Fourier component $n=4$, we find 11,600 rad/s; 77,300 rad/s; and 110,000 rad/s. We can now choose an increment Δt for the solution of our system of equations by using ω_{max} for $n=4$ in Eq. (50) after first having assigned a suitable value to the parameter μ.

With the chosen value for Δs, we find solutions to be unstable for values of $\Delta t \geq 0.225 \times 10^{-5}$ s and stable for values of $\Delta t \leq 0.200 \times 10^{-5}$ s. With the chosen Δs and values of $\Delta t \leq 0.200 \times 10^{-5}$ s, we may obtain, after solving our system of equations for each of the four Fourier components of loading, values of the primary and secondary variables by use of Eqs. (23) and (24).

We illustrate the results of our solution by showing for the meridian $\theta = 0$ values of $w(s_9,t)$ found by using both $\Delta t = 0.1500 \times 10^{-5}$ s and $\Delta t = 0.1875 \times 10^{-5}$ s in Table 1. To illustrate graphically the nature of the shell response, we show plots of $w(s_9,t)$ and $Q(s_0,t)$ in Figs. 5 and 6, respectively. It may be observed in comparing the results given herein with the corresponding results given in Refs. 1 and 3 that the overall agreement is reasonably good. We expect some differences between the results given herein and the results given in Ref. 1 due to the inclusion of circumferential inertia forces

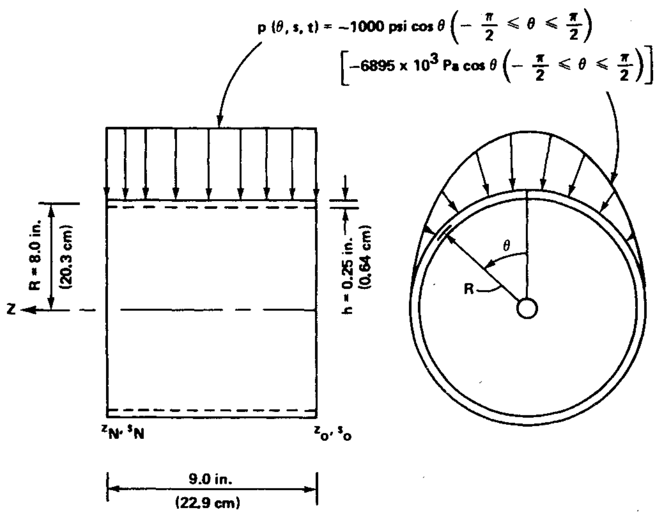

Fig. 4 Example cylindrical shell and loading.

in the present formulation, whereas these quantities were neglected in Ref. 1.

Solution times are not available now for the example case as solved in Refs. 1 and 3. Execution time for the solution given here was 213 s for $\Delta t = 0.15 \times 10^{-5}$ s and printing of the results at every one of the 220 time steps. The execution time was 125 s for $\Delta t = 0.1875 \times 10^{-5}$ s and printing of the results at every eighth time step of the 176 time steps.

Comparison with Other Formulations

The differential equations governing the response of a shell to time-dependent loadings may be given in various forms. Additionally, various choices are available for the finite-difference representations of the derivatives appearing in the differential equations. We expect values of Δt which produce numerically stable explicit solutions to the finite-difference equations to be dependent upon the formulation of the differential equations, the orders of both the spatial and temporal finite-difference representations, and the meridional increment Δs. For a given form of the differential equations

Table 1 Example solutions for $w(s_9,t)$ at $\theta = 0$ with $\Delta t = 0.15 \times 10^{-5}$ s and $\Delta t = 0.1875 \times 10^{-5}$ s

	$w(s_9,t)$, in. (2.54 cm)	
t, 10^{-5} s	$\Delta t = 0.15 \times 10^{-5}$ s	$\Delta t = 0.1875 \times 10^{-5}$ s
0	0.0	0.0
1.50	-5.9706×10^{-4}	-5.9706×10^{-4}
3.00	-2.2969×10^{-3}	-2.2969×10^{-3}
4.50	-4.8685×10^{-3}	-4.8685×10^{-3}
6.00	-8.0724×10^{-3}	-8.0736×10^{-3}
7.50	-1.1031×10^{-2}	-1.1029×10^{-2}
9.00	-1.3820×10^{-2}	-1.3821×10^{-2}
10.50	-1.6828×10^{-2}	-1.6828×10^{-2}
12.00	-1.9629×10^{-2}	-1.9630×10^{-2}
13.50	-2.2107×10^{-2}	-2.2108×10^{-2}
15.00	-2.3573×10^{-2}	-2.3573×10^{-2}
16.50	-2.3737×10^{-2}	-2.3738×10^{-2}
18.00	-2.2664×10^{-2}	-2.2664×10^{-2}
19.50	-2.0156×10^{-2}	-2.0157×10^{-2}
21.00	-1.6143×10^{-2}	-1.6137×10^{-2}
22.50	-1.1590×10^{-2}	-1.1592×10^{-2}
24.00	-7.2312×10^{-3}	-7.2267×10^{-3}
25.50	-3.4112×10^{-3}	-3.4134×10^{-3}
27.00	-5.9933×10^{-4}	-5.9882×10^{-4}
28.50	1.5511×10^{-3}	1.5551×10^{-3}
30.00	2.5163×10^{-3}	2.5126×10^{-3}
31.50	2.0787×10^{-3}	2.0801×10^{-3}
33.00	6.2636×10^{-4}	6.2437×10^{-4}

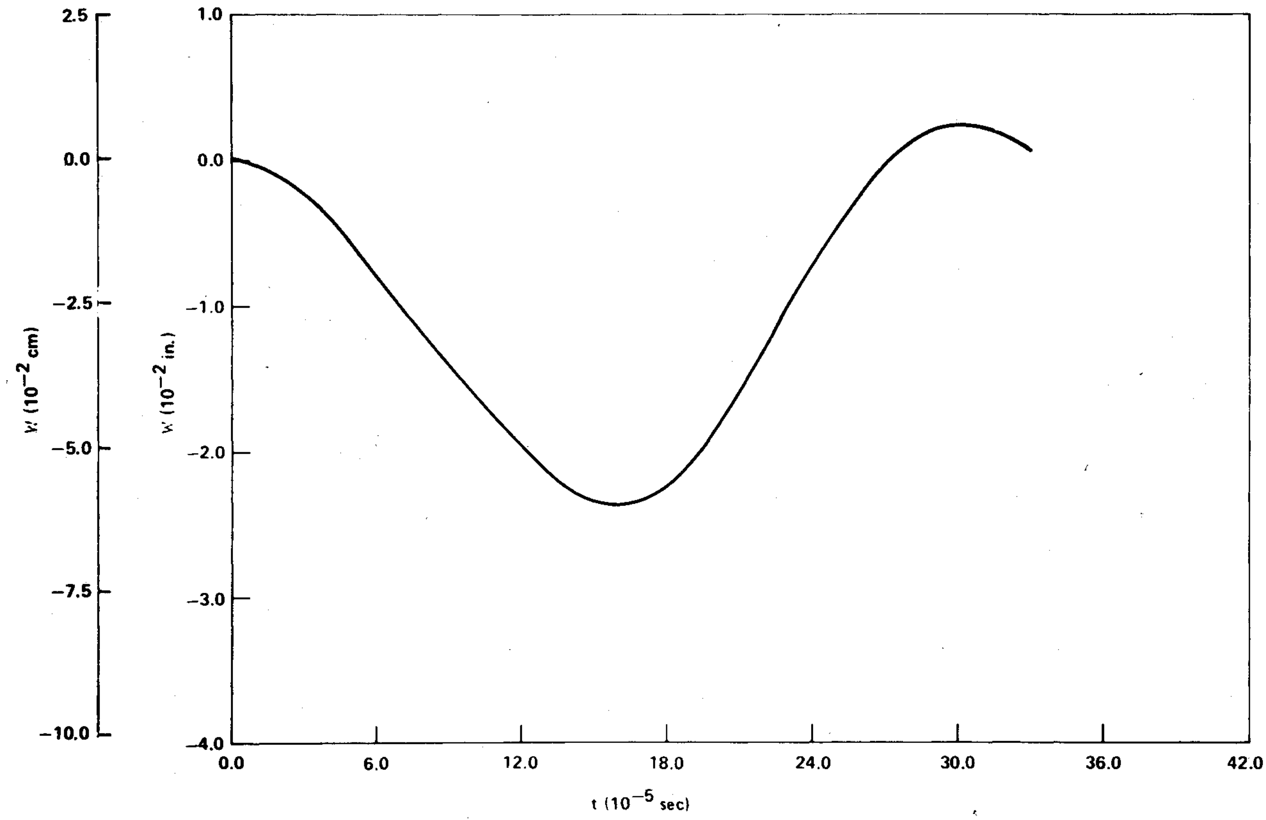

Fig. 5 Plot of $w(s_9,t)$ at $\theta=0$ for example solution.

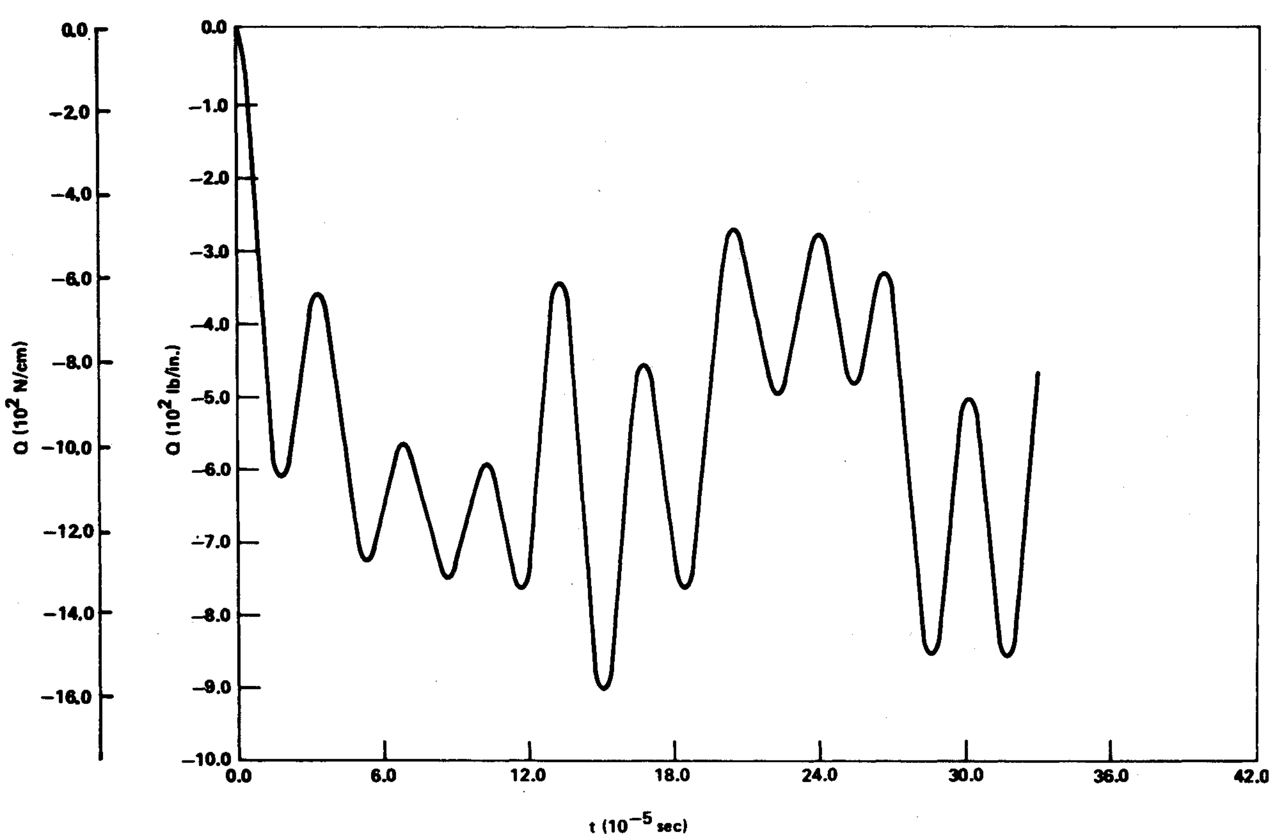

Fig. 6 Plot of $Q(s_9,t)$ at $\theta=0$ for example solution.

Table 2 Values of time increment for stable and unstable solutions for example case with different formulations, finite-difference representations, and meridional increments

Formulation	Spatial finite-difference representation	Meridional increment cm	in.	Time increment, 10^{-5} s Stable	Unstable
Three displacements	Ordinary	0.318	0.125	≤0.028125	≥0.05625
		0.635	0.250	≤0.1125	≥0.125
		1.270	0.500	≤0.225	≥0.250
	High order	0.318	0.125	≤0.010	≥0.050
		0.635	0.250	≤0.100	≥0.125
		1.270	0.500	≤0.200	≥0.225
Eight generalized forces and displacements	Ordinary	0.318	0.125	≤0.060	≥0.090
		0.635	0.250	≤0.150	≥0.200
		1.270	0.500	≤0.320	≥0.360
	High order	0.318	0.125	Unstable for all choices of meridional and time increments	
		0.635	0.250		
		1.270	0.500		

and a given meridional increment, we expect the accuracy of the static solution to improve with an increase in the order of the spatial finite-difference representations used for the derivatives. For values of Δt which produce numerically stable explicit solutions, this increase in accuracy will also manifest itself in the time-dependent problem. For a given formulation of the differential equations, a given order of spatial finite-difference representations, and a given Δs, we expect stable explicit solutions to require smaller values of Δt with an increase in the order of the temporal finite-difference representations. Thus, as reported in Ref. 2, an increase in the order of the temporal finite-difference representations necessitated a smaller time increment to obtain stable explicit solutions for the conical shell than that required for the ordinary temporal finite-difference representations. However, we make comparisons here of the explicit solution stability requirements for the formulation and spatial finite-difference representations given in Ref. 1, previously discussed relative to the work of Ref. 3, and given herein for ordinary temporal finite-difference representations only. To further our comparisons, stability requirements for the system of equations obtained by using ordinary spatial finite-difference representations in conjunction with the formulation of the differential equations used here were investigated for the axisymmetric case only. We make our comparisons for the typical example included here, which is the same example considered in Refs. 1 and 3. These comparisons are shown in Table 2 for three different choices of meridional increments.

It is seen in Table 2 that the use of ordinary spatial derivative representations permits the use of a somewhat larger time increment for either formulation of the equations than the time increment required for the high-order spatial derivative representations. And, as discussed previously relative to the work in Ref. 3, the use of higher-order spatial derivatives for the eight first-order differential equation formulation resulted in unstable solutions for all choices of time increments. It is also seen in Table 2 that considerably larger time increments may be used with the eight first-order differential equation formulation and ordinary spatial derivative representations than with the three higher-order differential equation formulation and ordinary spatial derivative representations. For the example case, however, solutions obtained for the three higher-order differential equation formulation and higher-order spatial derivative representations with a 0.50 in. meridional increment are essentially equivalent to those found for the eight first-order differential equation formulation and ordinary spatial derivative representations with a ¼-in. meridional increment. Thus, by using the formulation given here, we were able to increase the time increment from 0.15×10^{-5} s to 0.20×10^{-5} s and reduce the number of meridional points at which solutions were found by essentially 50%.

Conclusions

The results shown in Table 1 and in Figs. 5 and 6 for the typical example indicate that very satisfactory and accurate solutions may be found by the formulation and finite-difference representations which have been used here. By formulating our equations in terms of the transverse, meridional, and circumferential displacements as the dependent variables in our field equations, we have obtained for a high-order spatial derivative representation a system of finite-difference equations for which numerically stable explicit solutions may generally be found for a wide range of practical values of time and spatial increments. It is expected that the formulation and higher-order spatial derivative representations used here may generally make possible the choice of larger meridional and time increments than would usually be expected with the formulation and spatial derivative representations used in Ref. 1. We conclude that the finite-difference methods developed here constitute an efficient and satisfactory procedure for the analysis of rotationally symmetric general shells.

References

[1] Smith, T. A., "Numerical Analysis of Rotationally Symmetric Shells under Transient Loadings," *AIAA Journal*, Vol. 9, April 1971, pp. 637-643.

[2] Smith, T. A., *Static and Dynamic Analysis of Conical Shells under Symmetrical Loads*, Tech. Rept. RL-72-14, Nov. 1972, U.S. Army Missile Command, Redstone Arsenal, Ala.

[3] Smith, T. A., *Implicit High Order Finite Difference Analysis of Rotationally Symmetric Shells,* Tech. Rept. RL-73-9, Oct. 15, 1973, U.S. Army Missile Command, Redstone Arsenal, Ala.

[4] Reissner, E., "A New Derivation of the Equations for the Deformation of Elastic Shells," *American Journal of Mathematics*, Vol. 63, 1941, pp. 177-184.

[5] Smith, T. A., "Explicit High Order Finite Difference Analysis of Rotationally Symmetric Shells," Tech. Rept. TL-77-1, Feb. 2, 1977, U.S. Army Missile Research and Development Command, Redstone Arsenal, Ala.

[6] Kalnins, A., "Analysis of Shells of Revolution Subjected to Symmetrical and Nonsymmetrical Loads," *Transactions ASME 31E, Journal of Applied Mechanics*, Vol. 3, Sept. 1964, pp. 467-476.